Optimality conditions without continuity in multivalued optimization using approximations as generalized derivatives

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Abstract. We propose a notion of approximations as generalized derivatives for multivalued mappings and establish both necessary conditions and sufficient conditions of orders 1 and 2 for various kinds of efficiency in multivalued vector optimization without convexity and even continuity. Compactness assumptions are also relaxed. Our theorems include several recent existing results in the literature as special cases.

Key words. Approximations and strong approximations for multifunctions, first and second-order optimality conditions, weak efficiency, firm efficiency, asymptotic pointwise compactness

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1. Introduction and preliminaries

Differentiability assumptions are often crucial for a classical problem in all areas of continuous mathematics, since derivatives are local linear approximations for the involved nonlinear mappings and then supply a much simpler approximated linear problem, replacing the original nonlinear problem. However, such differentiability assumptions are too severe and not satisfied in many practical situations. Relaxing these assumptions has been one of the main idea in optimization for more than three decades now and constituted an important field of research called nonsmooth optimization. Most of contributions in this field are based on using generalized derivatives which are local approximations bearing not the whole linearity but still parts of linearity. A lot of notions of generalized derivatives have been proposed. Each of them is suitable for a class of problems. The Clarke derivative [5] is introduced for locally Lipschitz mappings; the quasidifferentiability of Demyanov and Rubinov [6] requires directionally differentiability to be defined; the approximate Jacobian proposed in [8] (later renamed as pseudo Jacobian) exists only for continuous mappings, etc. The approximations, introduced in [11] for order 1 and in [1] for order 2, is defined for general mappings which are even discontinuous. In this note we extend these definitions to the case of multifunctions.

The major goal for generalized derivatives to be proposed is establishing optimality conditions in nonsmooth optimization problems. We see from the very beginning of classical optimization that derivatives play a fundamental role in the Fermat theorem, the first necessary optimality condition. We would say that all generalized derivatives are used in similar ways as the classical derivative in the Fermat theorem. In the literature we observe only [11, 1, 2, 12-14, 16, 17] which deal with this kind of approximations as generalized derivatives. This notion was used in [11] to study metric regularity, in [1] for establishing second-order necessary optimality conditions in compactness case. Second-order approximations of scalar functions are used for support functions in [2] to scalarize vector problems so that second-order optimality conditions can be established, but under strict (first-order) differentiability and compactness assumptions. In [12-14] we used first and second-order approximations of single-valued mappings to derive first and second-order necessary conditions and sufficient conditions for various kinds of efficiency in nonsmooth vector optimization problems of several types.

In this paper we develop the results of our talk presented (but unpublished) at an international conference [16]. Namely, after extending the notion of first and second-order approximations of a mapping to the case of a multivalued mapping, we use this notion to establish both necessary conditions and sufficient conditions of both orders 1 and 2 for weak and firm efficiencies in multivalued vector optimization with set constraints, without continuity and convexity assumptions. In [17] we develop such optimality conditions also for proper efficiency and in problems with functional constraints. The problem under our consideration here is as follows. Throughout this paper, unless otherwise specified, let X and Y be normed spaces, Y being partially ordered by a convex cone C with nonempty interior, $S \subseteq X$ be a nonempty subset and $F : X \to 2^Y$ be a multifunction (i.e. a multivalued mapping). We are concerned with the problem

$$\text{(P)} \qquad \min F(x), \text{ subject to } x \in S.$$

Here "min" means minimizing, i.e. finding efficient solutions in the sense defined by the end of this section. The layout of the paper is as follows. In the rest of this section we recall definitions and preliminaries needed for our later investigation. Section 2 is devoted to defining first and second-order approximations of a multivalued mapping. In Section 3 we establish necessary conditions of order 1 for weak efficiency and sufficient conditions of order 1 for firm efficiency of problem (P). We develop such conditions for these kinds of efficiency, but of order 2, in the final Section 4.

Our notations are rather standard. $\mathbb{N} = \{1, 2, ...n, ...\}$ and $\|.\|$ stands for the norm in any normed space (the context will make it clear what space is concerned). B_X denotes the open unit ball in X and $B_X(x, r) = \{z \in X \mid \|x - z\| < r\}$; X^* is the topological dual of X with $\langle ., . \rangle$ being the canonical pairing. $L(X, Y)$ denotes the space of all bounded linear mappings from X into Y and $B(X, X, Y)$ that of all bounded bilinear mappings from $X \times X$ into Y. For a cone $C \subseteq X$, $C^* = \{x^* \in X^* \mid \langle x^*, c\rangle \geq 0, \forall c \in C\}$ is the positive polar cone of C. For $A \subseteq X$, int$A$, cl$A$ and bd$A$ denote the interior, closure and boundary of A, respectively. For $t > 0$ and $k \in \mathbb{N}$, $o(t^k)$ stands for a moving point (in a normed space) such that $o(t^k)/t^k \to 0$ as $t \to 0^+$. We will use the following tangent sets of $A \subseteq X$ at $x_0 \in A$.

(a) The contingent (or Bouligand) cone of A at x_0, see [3], is

$$T(A, x_0) = \{v \in X \mid \exists t_n \to 0^+, \exists v_n \to v, \forall n \in \mathbb{N}, x_0 + t_n v_n \in A\}.$$

(b) The second-order contingent set of A at (x_0, v), see [3], is

$$T^2(A, x_0, v) = \{w \in X \mid \exists t_n \to 0^+, \exists w_n \to w, \forall n \in \mathbb{N}, x_0 + t_n v + \frac{1}{2}t_n^2 w_n \in A\}.$$

(c) the asymptotic second-order tangent cone of A at (x_0, v), see [4, 18], is

$$T^{''}(A, x_0, v) = \{w \in X \mid \exists (t_n, r_n) \to (0^+, 0^+) : \frac{t_n}{r_n} \to 0, \exists w_n \to w.$$

$$\forall n \in \mathbb{N}, x_0 + t_n v + \tfrac{1}{2}t_n r_n w_n \in A\}.$$

Lemma 1.1 [10]. Assume that X is a finite dimensional space $\mathbb{R}^m$ and $x_0 \in A \subseteq X$. If $x_n \in A \setminus \{x_0\}$ tends to x_0, then there exists $v \in T(A, x_0) \setminus \{0\}$ and a subsequence, denoted again by x_n, such that, for $t_n = \|x_n - x_0\|$,

(i) $\frac{1}{t_n}(x_n - x_0) \to v$;

(ii) either $z \in T^2(A, x_0, v) \cap v^{\perp}$ exists such that $(x_n - x_0 - t_n v)/\frac{1}{2}t_n^2 \to z$ or $z \in T^{''}(A, x_0, v) \cap v^{\perp} \setminus \{0\}$ and $r_n \to 0^+$ with $\frac{t_n}{r_n} \to 0^+$ exist such that $(x_n - x_0 - t_n v)/\frac{1}{2}t_n r_n \to z$, where $v^{\perp} = \{y \in \mathbb{R}^m \mid \langle y, v\rangle = 0\}$.

Recall now notions of efficiency in vector optimization. Consider a subset V of the objective space Y. A point $y_0 \in V$ is called an efficient point (weak efficient point, strict efficient point, respectively) of V if

$$(V - y_0) \cap -C \subseteq (-C) \cap C$$

$$((V - y_0) \cap -\text{int}C = \emptyset,$$

$$(V - y_0) \cap (-C \setminus \{0\}) = \emptyset, \quad \text{respectively}).$$

The set of efficient and weak efficient points are denoted by $\text{Min}_C V$, $\text{WMin}_C V$ and $\text{StrMin}_C V$, respectively. Apply now these notions to problem (P). A point (x_0, y_0) with $x_0 \in S$ and $y_0 \in F(x_0)$ is said to be a local weak efficient solution of (P) if there is a neighborhood U of x_0 such that, $\forall x \in S \cap U$,

$$(F(x) - y_0) \cap - \text{int}C = \emptyset, \tag{1}$$

while (x_0, y_0) is called a local efficient solution if (1) is replaced by

$$(F(x) - y_0) \cap -C \subseteq (-C) \cap C.$$

We extend the firm efficiency notion, see [9, 15], to the case of multivalued optimization as follows.

Definition 1.1. Let $x_0 \in S$, $y_0 \in F(x_0)$ and $m \in \mathbb{N}$. Then (x_0, y_0) is said to be a local firm efficient solution of order m if there are a neighborhood U of x_0 and $\gamma > 0$ such that $y_0 \in \text{StrMin}_C F(x_0)$ and, for all $x \in S \cap U \setminus \{x_0\}$,

$$(F(x) - y_0) \cap (B_Y(0, \gamma \|x - x_0\|^m) - C) = \emptyset.$$

In the sequel let LWE(P), LE(P) and LFE(m,P) stand for the sets of the local weakly efficient solutions, of the local efficient solutions and of the local firm efficient solutions of order m, respectively, of problem (P). Then it is clear that, for $p, m \in \mathbb{N}$ with $p \geq m$,

$$\text{LFE}(m, \text{P}) \subseteq \text{LFE}(p, \text{P}) \subseteq \text{LE(P)} \subseteq \text{LWE(P)}.$$

Hence, necessary conditions for the right-most term are valid also for the others and sufficient conditions for the left-most term hold true for the others as well.

For a multifunction $H : X \to 2^Y$ the domain of H is

$$\text{dom}H = \{x \in X \mid H(x) \neq \emptyset\}.$$

H is said to be upper semicontinuous (usc) at $x_0 \in \text{dom}H$ if for all open set $V \supseteq H(x_0)$, there is a neighborhood U of x_0 such that $V \supseteq H(U)$. H is termed lower semicontinuous (lsc) at $x_0 \in \text{dom}H$ if for all open set $V \cap H(x_0) \neq \emptyset$, there is a neighborhood U of x_0 such that for all $x \in U$, $V \cap H(x) \neq \emptyset$.

2. First and second-order approximations of multifunctions

Consider a multifunction $H : X \to 2^Y$, $x_0 \in \text{dom}H$ and $y_0 \in F(x_0)$.

Definition 2.1

(i) A subset $A_H(x_0, y_0)$ of $L(X, Y)$ is said to be a first-order approximation of H at (x_0, y_0) if there exists a neighborhood U of x_0 such that, for all $x \in U \cap$ dom$H$, there are positive r_x with $r_x \|x - x_0\|^{-1} \to 0^+$ and $y \in H(x)$ satisfying

$$y - y_0 \in A_H(x_0, y_0)(x - x_0) + r_x B_Y.$$

(ii) A subset $A_H^S(x_0, y_0)$ of $L(X, Y)$ is called a first-order strong approximation of H at (x_0, y_0) if there exists a neighborhood U of x_0 such that, for all $x \in U \cap$ dom$H$, there is positive r_x with $r_x \|x - x_0\|^{-1} \to 0^+$ such that, for all $y \in H(x)$,

$$y - y_0 \in A_H^S(x_0, y_0)(x - x_0) + r_x B_Y.$$

(iii) A pair $(A_H(x_0,y_0), B_H(x_0,y_0))$, where $A_H(x_0,y_0) \subseteq L(X,Y)$ and $B_H(x_0,y_0) \subseteq B(X,X,Y)$, is called a second-order approximation of H at (x_0,y_0) if $A_H(x_0,y_0)$ is a first-order approximation of H at (x_0,y_0) and there is a neighborhood U of x_0 such that, for all $x \in U\cap$ dom$H$, there are positive r_x^2 with $r_x^2\|x-x_0\|^{-2} \to 0^+$ and $y \in H(x)$ satisfying

$$y - y_0 \in A_H(x_0,y_0)(x-x_0) + B_H(x_0,y_0)(x-x_0,x-x_0) + r_x^2 B_Y.$$

(iv) A pair $(A_H^S(x_0,y_0), B_H^S(x_0,y_0))$, where $A_H^S(x_0,y_0) \subseteq L(X,Y)$ and $B_H^S(x_0,y_0) \subseteq B(X,X,Y)$, is termed a second-order strong approximation of H at (x_0,y_0) if $A_H^S(x_0,y_0)$ is a first-order strong approximation of H at (x_0,y_0) and there is a neighborhood U of x_0 such that, for all $x \in U\cap$ dom$H$, there exists positive r_x^2 with $r_x^2\|x-x_0\|^{-2} \to 0^+$ such that, for all $y \in H(x)$,

$$y - y_0 \in A_H^S(x_0,y_0)(x-x_0) + B_H^S(x_0,y_0)(x-x_0,x-x_0) + r_x^2 B_Y.$$

In this paper we will impose on these approximations the following relaxed compactness.

Definition 2.2

(i) Let M_n and M be in $L(X,Y)$. The sequence M_n is said to pointwise converge to M and written as $M_n \xrightarrow{\text{p}} M$ or $M = \text{p-lim}\, M_n$ if $\lim M_n(x) = M(x)$ for all $x \in X$. A similar definition is adopted for $N_n, N \in B(X,X,Y)$.

(ii) A subset $A \subseteq L(X,Y)$ ($B \subseteq B(X,X,Y)$, respectively) is called (sequentially) asymptotically pointwise compact, or (sequentially) asymptotically p-compact if

(a) each norm bounded sequence $\{M_n\} \subseteq A$ ($\subseteq B$, respectively) has a subsequence $\{M_{n_k}\}$ and $M \in L(X,Y)$ ($M \in B(X,X,Y)$, respectively) such that $M = \text{p-lim}\, M_{n_k}$,

(b) for each sequence $\{M_n\} \subseteq A$ ($\subseteq B$, respectively) with $\lim \|M_n\| = \infty$, the sequence $\{M_n/\|M_n\|\}$ has a subsequence which pointwise converges to some $M \in L(X,Y) \setminus \{0\}$ ($M \in B(X,X,Y) \setminus \{0\}$, respectively).

(iii) If in (ii), pointwise convergence, i.e. p-lim, is replaced by convergence, i.e. lim, a subset $A \subseteq L(X,Y)$ (or $B \subseteq B(X,X,Y)$) is called (sequentially) asymptotically compact.

Since only sequential convergence is met in this paper, we will omit the word "sequentially". For $A \subseteq L(X,Y)$ and $B \subseteq B(X,X,Y)$ we adopt the notations:

$$\text{p-cl}\, A = \{M \in L(X,Y) : \exists (M_n) \subseteq A, M = \text{p-lim}\, M_n\}, \tag{2}$$

$$\text{p-cl}\, B = \{N \in B(X,X,Y) : \exists (N_n) \subseteq B, N = \text{p-lim}\, N_n\}, \tag{3}$$

$$A_\infty = \{M \in L(X,Y) : \exists (M_n) \subseteq A, \exists t_n \to 0^+, M = \lim t_n M_n\}, \tag{4}$$

$$\text{p-}A_\infty = \{M \in L(X,Y) : \exists (M_n) \subseteq A, \exists t_n \to 0^+, M = \text{p-lim}\, t_n M_n\}, \tag{5}$$

$$\text{p-}B_\infty = \{N \in B(X,X,Y) : \exists (N_n) \subseteq B, \exists t_n \to 0^+, N = \text{p-lim}\, t_n N_n\}. \tag{6}$$

The sets (2), (3) are pointwise closures; (4) is just the definition of the recession cone of A. So (5), (6) are pointwise recession cones.

Remark 2.1

(i) If X is finite dimensional, a convergence occurs if and only if the corresponding pointwise convergence does, but in general the "if" does not hold, see [12, Example 3.1].

(ii) If X and Y are finite dimensional, every subset is asymptotically p-compact and asymptotically compact but in general the asymptotical compactness is stronger, as shown by [12, Example 3.2].

(iii) Assume that $\{M_n\} \subseteq L(X,Y)$ is norm bounded. If $x_n \to x$ in X and $M_n \xrightarrow{\text{p}} M$ in $L(X,Y)$, then $M_n x_n \to Mx$ in Y. Similarly, if $x_n \to x$, $y_n \to y$ in X, $N_n \xrightarrow{\text{p}} N$ in $B(X,X,Y)$ and $\{N_n\}$ is norm bounded then $N_n(x_n,y_n) \to N(x,y)$ in Y.

Indeed, the conclusion is derived from the following evaluations

$$\|M_n x_n - Mx\| \le \|M_n x_n - M_n x\| + \|M_n x - Mx\| \le \|M_n\|\|x_n - x\| + \|M_n x - Mx\|;$$

$$\|N_n(x_n, y_n) - N(x, y)\| \le \|N_n(x_n, y_n) - N_n(x_n, y)\| + \|N_n(x_n, y) - N_n(x, y)\| +$$
$$\|N_n(x, y) - N(x, y)\| \le \|N_n\|\|x_n\|\|y_n - y\| + \|N_n\|\|x_n - x\|\|y\| + \|N_n(x, y) - N(x, y)\|.$$

The following example gives a multivalued map F, which is neither usc nor lsc at x_0, but has even second-order strong approximations.

Example 2.1. Let $F : \mathbb{R} \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y \ge \sqrt{x}\} & \text{if } x > 0, \\ \{y \in \mathbb{R} \mid y \le \frac{1}{x}\} & \text{if } x < 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

Let $(x_0, y_0) = (0, 0)$. Then F is neither usc nor lsc at x_0 but F has the following approximations, for fixed positive α and $\beta > 0$,

$$A_F(x_0, y_0) = (\alpha, +\infty), \;\; A_F^S(x_0, y_0) = (\beta, +\infty),$$

$$B_F(x_0, y_0) = B_F^S(x_0, y_0) = \{0\}.$$

In the next example F is not usc at x_0 but $A_F(x_0, y_0)$ is even a singleton.

Example 2.2. Let $F : \mathbb{R}^2 \to 2^{\mathbb{R}}$ be defined by

$$F(x_1, x_2) = \begin{cases} \{y \in \mathbb{R} \mid \frac{2}{3}|x_1|^{\frac{3}{2}} + x_2^2 \le y \le \frac{1}{|x_1|+|x_2|}\} & \text{if } (x_1, x_2) \ne (0, 0), \\ \{0\} & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

Then F is not usc at $x^0 = (0, 0)$. But for $y_0 = 0$ we have

$$A_F(x^0, y_0) = \{0\},$$

$$A_F^S(x^0, y_0) = (\mathbb{R} \setminus \{0\}) \times \{0\} \cup \{0\} \times (\mathbb{R} \setminus \{0\}),$$

$$B_F(x^0, y_0) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha > 1 \right\},$$

$$B_F^S(x^0, y_0) = \{0\}.$$

Note that a similar example for a single-valued mapping does not exist, since a single-valued mapping has a first-order approximation at x_0 being a singleton if and only if it is Fréchet differentiable at x_0 and hence continuous at this point.

3. First-order optimality conditions

Theorem 3.1 (Necessary condition). Consider problem (P). Assume that $A_F(x_0, y_0)$ is an asymptotically p-compact first-order approximation of F at (x_0, y_0). If $(x_0, y_0) \in$ LWE(P) then, for each $v \in T(S, x_0)$ there is $M \in \text{p-cl}A_F(x_0, y_0) \bigcup (\text{p-}A_F(x_0, y_0)_\infty \setminus \{0\})$ such that

$$Mv \notin -\text{int}C.$$

Proof. Let $v \in T(S, x_0)$ be arbitrary and fixed. By the definition of a contingent cone, there is $(t_n, v_n) \to (0^+, v)$ such that $x_0 + t_n v_n \in S$ for all $n \in \mathbb{N}$. By the weak efficiency of (x_0, y_0) one has, for large n and all $y \in F(x_0 + t_n v_n)$,

$$y - y_0 \notin -\text{int}C.$$

On the other hand, as $A_F(x_0, y_0)$ is a first-order approximation, there are positive r_n with $r_n t^{-1} \to 0^+$ and $y_n \in F(x_0 + t_n v_n)$ such that

$$y_n - y_0 \in A_F(x_0, y_0)(t_n v_n) + r_n B_Y.$$

Therefore, $M_n \in A_F(x_0, y_0)$ and $\bar{y}_n \in r_n B_Y$ exist such that

$$M_n(t_n v_n) + \bar{y}_n \notin -\text{ int } C. \tag{7}$$

If $\{M_n\}$ is norm bounded, one can assume that $M_n \xrightarrow{\text{p}} M \in$ p-cl$A_F(x_0, y_0)$. Dividing (7) by t_n and passing to limit one gets $Mv \notin -$ int C. If $\{M_n\}$ is unbounded, one can assume that $\|M_n\| \to \infty$ and $\frac{M_n}{\|M_n\|} \xrightarrow{\text{p}} M \in$p-$A_F(x_0, y_0)_\infty \setminus \{0\}$. Dividing (7) by $\|M_n\| t_n$ one obtains in the limit $Mv \notin -$ int C. □

If F is single-valued, Theorem 3.1 collapses to Theorem 3.3 of [13], which was shown there to improve or include many exising results. The following example shows that, for F being multivalued, Theorem 3.1 is easy to be applied.

Example 3.1. Let $X = Y = \mathbb{R}$, $S = [0, +\infty)$, $C = \mathbb{R}_+$, $x_0 = y_0 = 0$ and

$$F(x) = \begin{cases} \{y \in \mathbb{R} \mid y \leq \frac{1}{\sqrt[3]{x}}\} & \text{if } x > 0, \\ \{y \in \mathbb{R} \mid y \geq \sqrt[3]{-x}\} & \text{if } x < 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

Then $T(S, x_0) = S$ and for a fixed $\alpha < 0$ we have $A_F(x_0, y_0) = (-\infty, \alpha)$, cl$A_F(x_0, y_0) = (-\infty, \alpha]$, $A_F(x_0, y_0)_\infty = (-\infty, 0]$. Taking $v = 1 \in T(S, x_0)$ one sees that, for all $M \in$ cl$A_F(x_0, y_0) \cup (A_F(x_0, y_0)_\infty \setminus \{0\} = (-\infty, 0)$,

$$Mv = M \in -\text{ int } C.$$

Due to Theorem 3.1, (x_0, y_0) is not a local (weakly efficient) solution of problem (P).

Theorem 3.2 (Sufficient condition). Consider problem (P) with X being finite dimensional. Assume that $A_F^S(x_0, y_0)$ is an asymptotically p-compact first-order strong approximation of F at (x_0, y_0), $x_0 \in S$ and $y_0 \in \text{StrMin}_C F(x_0)$. Impose further that, for all $v \in T(S, x_0) \setminus \{0\}$ and all $M \in$ p-cl$A_F^S(x_0, y_0) \bigcup($p-$A_F^S(x_0, y_0)_\infty \setminus \{0\})$, one has

$$Mv \notin -\text{cl}C.$$

Then $(x_0, y_0) \in$ LFE(1, P).

Proof. Reasoning ad absurdum, suppose the existence of $x_n \in S \cap B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ such that, for each $n \in \mathbb{N}$, there is $y_n \in F(x_n)$ such that

$$y_n - y_0 \in B_Y(0, \frac{1}{n}\|x_n - x_0\|) - C.$$

As X is finite dimensional, we can assume that $\frac{x_n - x_0}{\|x_n - x_0\|}$ tends to a point v in $T(S, x_0) \setminus \{0\}$. On the other hand, for large n there is positive r_n with $r_n \|x_n - x_0\|^{-1} \to 0^+$ such that

$$y_n - y_0 \in A_F^S(x_0, y_0)(x_n - x_0) + r_n B_Y.$$

Hence, there are $M_n \in A_F^S(x_0, y_0)$ and $\bar{y}_n \in r_n B_Y$ such that

$$M_n(x_n - x_0) + \bar{y}_n \in B_Y(0, \frac{1}{n}\|x_n - x_0\|) - C.$$

Arguing similarly as in the final part of the proof of Theorem 3.1, we obtain $M \in$ p-cl$A_F^S(x_0, y_0) \bigcup($p-$A_F^S(x_0, y_0)_\infty \setminus \{0\})$ such that $Mv \in -$ cl C, a contradiction. □

Theorem 3.2 includes Theorem 3.4 of [13] as a special case where F is single-valued. The following example explains how to employ Theorem 3.2.

Example 3.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, +\infty)$, $C = \mathbb{R}_+^2$, $x_0 = 0$, $(y_0, z_0) = (0, 0) \in Y$ and

$$F(x) = \begin{cases} \{(y, z) \in \mathbb{R}^2 \mid y \geq \sqrt[3]{x}, z = x\} & \text{if } x > 0, \\ \{(0, 0)\} & \text{if } x = 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

Then $(y_0, z_0) \in \text{StrMin}_C F(x_0)$ and for any fixed $\alpha > 0$ we can take a strong approximation as follows

$$A_F^S(x_0, (y_0, z_0)) = \{(y, z) \in \mathbb{R}^2 \mid y > \alpha, z = 1\},$$

$$\text{cl}A_F^S(x_0, (y_0, z_0)) = \{(y, z) \in \mathbb{R}^2 \mid y \geq \alpha, z = 1\},$$

$$A_F^S(x_0, (y_0, z_0))_\infty = \{(y, z) \in \mathbb{R}^2 \mid y \geq 0, z = 0\}.$$

It is clear that, $\forall v \in T(S, x_0) \setminus \{0\} = (0, +\infty)$, one has, $\forall M \in \text{cl}A_F^S(x_0, (y_0, z_0))$, $Mv = (yv, v) \notin -C$. Furthermore, $Mv = (yv, 0) \notin -C$ for all $M \in A_F^S(x_0, (y_0, z_0))_\infty \setminus \{0\}$. By Theorem 3.2, $(x_0, (y_0, z_0)) \in \text{LFE}(1, \text{P})$.

4. Second-order optimality conditions

Theorem 4.1 (Necessary condition). For problem (P) assume that $(A_F(x_0, y_0), B_F(x_0, y_0))$ is an asymptotically p-compact second-order approximation of F at (x_0, y_0) with $A_F(x_0, y_0)$ being norm bounded. Assume further that $(x_0, y_0) \in \text{LWE(P)}$. Then

(i) for all $v \in T(S, x_0)$, there exists $M \in \text{p-cl}A_F(x_0, y_0)$ such that $Mv \notin -\text{int } C$;

(ii) for all $v \in T(S, x_0)$ with $A_F(x_0, y_0)v \subseteq -\text{bd } C$ one has

(a) for each $w \in T^2(S, x_0, v)$, either $\overline{M} \in \text{p-cl}A_F(x_0, y_0)$ and $\overline{N} \in \text{p-cl}B_F(x_0, y_0)$ exist such that

$$\overline{M}w + 2\overline{N}(v, v) \notin -\text{ int } C,$$

or there is $\overline{N} \in \text{p-}B_F(x_0, y_0)_\infty \setminus \{0\}$ satisfying

$$\overline{N}(v, v) \notin -\text{int}C;$$

(b) for each $w \in T^{''}(S, x_0, v)$, either $M^{'} \in \text{p-cl}A_F(x_0, y_0)$ and $N^{'} \in \text{p-}B_F(x_0, y_0)_\infty$ exist such that

$$M^{'}w + N^{'}(v, v) \notin -\text{ int}C,$$

or one has $N^{'} \in \text{p-}B_F(x_0, y_0)_\infty \setminus \{0\}$ with

$$N^{'}(v, v) \notin -\text{int}C.$$

Proof. (i) This assertion follows from Theorem 3.1.

(ii) (a) Let $v \in T(S, x_0)$ with $A_F(x_0, y_0)(v) \subseteq -\text{bd } C$ and $w \in T^2(S, x_0, v)$. Then, there are $x_n \in S$ and $t_n \to 0^+$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n^2 \to w.$$

By the definition of the second-order approximation, there are $M_n \in A_F(x_0, y_0)$, $N_n \in B_F(x_0, y_0)$ and $o(\|x_n - x_0\|^2)$ such that, for large n,

$$M_n(x_n - x_0) + N_n(x_n - x_0, x_n - x_0) + o(\|x_n - x_0\|^2) \in F(x_n) - y_0.$$

The weak efficiency of (x_0, y_0) implies then, for some $o(t_n^2) \in Y$,

$$M_n w_n + 2N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n^2 \notin -\text{int } C. \tag{8}$$

We can assume that $M_n \xrightarrow{\text{p}} \overline{M}$ for some $\overline{M} \in \text{p-cl}A_F(x_0, y_0)$. If $\{N_n\}$ is norm bounded then $N_n \xrightarrow{\text{p}} \overline{N}$ for some $\overline{N} \in \text{p-cl}B_F(x_0, y_0)$. From (8) we get in the limit

$$\overline{M}w + 2\overline{N}(v, v) \notin -\text{ int } C.$$

If $\{N_n\}$ is unbounded, we can assume $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} \overline{N}$ for some $\overline{N} \in \text{p-}B_f(x_0)_\infty \setminus \{0\}$. Dividing (8) by $\|N_n\|$ and passing to limit gives $\overline{N}(v, v) \notin -\text{int } C$.

(b) For any $w \in T^{''}(S, x_0, v)$, there are $x_n \in S$ and $(t_n, r_n) \to (0^+, 0^+)$ with $\frac{t_n}{r_n} \to 0^+$ such that

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w.$$

Similarly as in (a) we have $M_n^{'}$ and $N_n^{'}$ satisfying the following relation, corresponding to (8),

$$M_n^{'} w_n + (\tfrac{2t_n}{r_n})N_n^{'}(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n r_n \notin -\text{int } C. \tag{9}$$

We can assume that $M_n^{'} \xrightarrow{\text{p}} M^{'} \in \text{p-cl}A_F(x_0, y_0)$. There are three possibilities.

(α) $(\frac{2t_n}{r_n})N_n^{'} \to 0$. From (9) we get in the limit

$$M^{'} w \notin -\text{int } C.$$

(β) If $(\frac{2t_n}{r_n})\|N_n^{'}\| \to a > 0$, then $\|N_n\| \to \infty$ and we can assume that $\frac{N_n^{'}}{\|N_n^{'}\|} \xrightarrow{\text{p}} N^{'} \in$ p-$B_F(x_0, y_0)_\infty \setminus \{0\}$. Passing (9) to limit yields

$$M^{'} w + aN^{'}(v, v) \notin -\text{ int } C.$$

(γ) If $(\frac{2t_n}{r_n})\|N_n^{'}\| \to \infty$, then dividing (9) by $(\frac{2t_n}{r_n})\|N_n^{'}\|$ and passing to limit gives

$$N^{'}(v, v) \notin -\text{int } C.$$

□

If F is single-valued, Theorem 4.1 collapses to Theorem 4.10 of [13]. The example below gives an application of Theorem 4.1 to a multivalued case.

Example 4.1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $S = \{(x, z) \in \mathbb{R}^2 \mid z = |x|^{\frac{3}{2}}\}$, $C = \mathbb{R}_+$, $(x_0, z_0) = (0, 0) \in X$, $y_0 = 0 \in Y$ and

$$F(x, z) = \begin{cases} \{y \in \mathbb{R} \mid -\frac{2}{3}|x|^{\frac{3}{2}} + z^2 - z \le y \le \frac{1}{x^2+z^2}\} & \text{if } (x, z) \neq (0, 0) \\ \{0\} & \text{if } (x, z) = (0, 0). \end{cases}$$

Then, for a fixed $\alpha < 0$,

$$T(S, (x_0, z_0)) = \{(x, z) \in \mathbb{R}^2 \mid z = 0\},$$
$$A_F((x_0, z_0), y_0) = \{(0, -1)\},$$
$$B_F((x_0, z_0), y_0) = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mid t < \alpha \right\},$$
$$\text{cl}B_F((x_0, z_0), y_0) = \left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mid t \le \alpha \right\},$$
$$B_F((x_0, z_0), y_0)_\infty = \left\{ \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \mid t \le 0 \right\}.$$

Taking $v = (1, 0) \in T(S, (x_0, z_0))$ one has

$$A_F((x_0, z_0), y_0)v = \{0\} \subseteq -\text{bd}C,$$
$$T^2(S, (x_0, z_0), v) = \emptyset,$$
$$T^{''}(S, (x_0, z_0), v) = \mathbb{R} \times \mathbb{R}_+.$$

Hence, for $w = (0, 1) \in T^{''}(S, (x_0, z_0), v)$ one obtains

$$(0, -1)w + N(v, v) = -1 + t < 0$$

for all $N \in B_F((x_0, z_0), y_0)_\infty$ and

$$N(v, v) = t < 0$$

for all $N \in B_F((x_0, z_0), y_0)_\infty \setminus \{0\}$. Taking into account Theorem 4.1, one sees that $((x_0, z_0), y_0)$ is not a local weakly efficient solution of problem (P) in this case.

Theorem 4.2 (Sufficient condition). Consider problem (P) with X being finite dimensional. Assume that $x_0 \in S$ and $y_0 \in \text{Min}_C F(x_0)$. Assume further that $(A_F^S(x_0, y_0), B_F^S(x_0, y_0))$ is an asymptotically p-compact second-order strong approximation of F at (x_0, y_0) with $A_F^S(x_0, y_0)$ being norm bounded. Then $(x_0, y_0) \in \text{LFE}(2, \text{P})$ if

(i) for all $v \in T(S, x_0) \setminus \{0\}$, $A_F^S(x_0, y_0)v \subseteq \text{cl } C$;

(ii) for each $v \in T(S, x_0) \setminus \{0\}$ with $\overline{M}v \in -\text{cl } C$ for some $\overline{M} \in \text{p-cl}A_F^S(x_0, y_0)$, for each

$N \in$ p-$B_F^S(x_0, y_0)_\infty \setminus \{0\}$, one has $N(v, v) \notin -\text{cl } C$ and

(a) $\forall w \in T^2(S, x_0, v) \cap v^\perp$, $\forall M \in$ p-cl$A_F^S(x_0, y_0)$, $\forall N \in$ p-cl$B_F^S(x_0, y_0)$,

$$Mw + 2N(v, v) \notin -\text{cl } C,$$

(b) $\forall w \in T''(S, x_0, v) \cap v^\perp \setminus \{0\}$, $\forall M \in$ p-cl$A_F^S(x_0, y_0)$, $\forall N \in$ p-$B_F^S(x_0, y_0)_\infty$,

$$Mw + N(v, v) \notin -\text{cl } C.$$

Proof. Suppose to the contrary that $x_n \in S \cap B_X(x_0, \frac{1}{n}) \setminus \{x_0\}$ exists such that

$$(F(x_n) - y_0) \cap (B_Y(0, \tfrac{1}{n}t_n^2) - C) \neq \emptyset, \tag{10}$$

where $t_n = \|x_n - x_0\|$. We can assume that $\frac{1}{t_n}(x_n - x_0) \to v \in T(S, x_0) \setminus \{0\}$. By (10) and by the definition of first-order strong approximations, for large n, there exist $\overline{M}_n \in A_F^S(x_0, y_0)$ and $o(t_n)$ such that

$$\overline{M}_n(x_n - x_0) + o(t_n) \in B_Y(0, \tfrac{1}{n}t_n^2) - C. \tag{11}$$

The norm boundedness of $A_F^S(x_0, y_0)$ allows to assume that $\overline{M}_n \xrightarrow{\text{p}} \overline{M} \in$ p-cl$A_F^S(x_0, y_0)$. Dividing (11) by t_n we get, in the limit, $\overline{M}v \in -\text{cl } C$. According to Lemma 1.1, there are only the following two possibilities.

(α) One has $w_n := (x_n - x_0 - t_n v)/\frac{1}{2}t_n^2 \to w \in T^2(S, x_0, v) \cap v^\perp$. By the definition of the second-order strong approximation, (10) implies the existence of $M_n \in A_F^S(x_0, y_0)$, $N_n \in B_F^S(x_0, y_0)$ and $o(\|x_n - x_0\|^2)$ such that, for large n,

$$M_n(x_n - x_0) + N_n(x_n - x_0, x_n - x_0) + o(\|x_n - x_0\|^2) \in B_Y(0, \tfrac{1}{n}t_n^2) - C.$$

This can be rewritten as

$$M_n w_n + 2N_n(v + \tfrac{1}{2}t_n w_n, v + \tfrac{1}{2}t_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n^2 = d_n/\tfrac{1}{2}t_n^2 - c_n', \tag{12}$$

where $d_n \in B_Y(0, \frac{1}{n}t_n^2)$ and $c_n' = (c_n + t_n M_n v)/\frac{1}{2}t_n^2 \in \text{cl } C$, since $c_n \in C$ and $A_F^S(x_0, y_0)v \subseteq \text{cl } C$. We can assume that $M_n \xrightarrow{\text{p}} M \in$ p-cl$A_F^S(x_0, y_0)$. If $\{N_n\}$ is norm bounded, we can assume further that $N_n \xrightarrow{\text{p}} N \in$ p-cl$B_F^S(x_0, y_0)$. In the limit (12) gives the contradiction

$$Mw + 2N(v, v) \in -\text{cl } C.$$

If $\{N_n\}$ is unbounded, we can assume that $\|N_n\| \to \infty$ and $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in$ p-$B_F(x_0, y_0)_\infty \setminus \{0\}$. We divide (12) by $\|N_n\|$ and pass it to limit to get $N(v, v) \in -\text{cl } C$, also a contradiction.

(β) There is $r_n \to 0^+$ such that $\frac{t_n}{r_n} \to 0^+$ and

$$w_n := (x_n - x_0 - t_n v)/\tfrac{1}{2}t_n r_n \to w \in T''(S, x_0, v) \cap v^\perp \setminus \{0\}.$$

Similarly as for the case (α), there are $M_n \in A_F^S(x_0, y_0)$, $N_n \in B_F^S(x_0, y_0)$ and $o(t_n^2)$ such that, for large n,

$$M_n w_n + (\tfrac{2t_n}{r_n})N_n(v + \tfrac{1}{2}r_n w_n, v + \tfrac{1}{2}r_n w_n) + o(t_n^2)/\tfrac{1}{2}t_n r_n = d_n/\tfrac{1}{2}t_n r_n - c_n', \tag{13}$$

where $d_n \in B_Y(0, \frac{1}{n}t_n^2)$ and $c_n' = (c_n + t_n M_n v)/\frac{1}{2}t_n r_n \in \text{cl } C$. We can assume that $M_n \xrightarrow{\text{p}} M \in$ p-cl$A_F^S(x_0, y_0)$. There are three subcases as follows.

- $(\frac{2t_n}{r_n})N_n \to 0$. Passing (13) to limit one gets $Mw \in -\text{cl } C$, contradicting assumption (ii) (b) (with $N = 0 \in$ p-$B_F^S(x_0, y_0)_\infty$).

- $(\frac{2t_n}{r_n})\|N_n\| \to a > 0$. Then $\|N_n\| \to \infty$ and we can assume that $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in$ p-$B_F(x_0, y_0)_\infty \setminus \{0\}$. Dividing (13) by $(\frac{2t_n}{r_n})\|N_n\|$ and passing to limit we obtain the contradiction

$$Mw + aN(v, v) \in -\text{cl } C.$$

- $(\frac{2t_n}{r_n})\|N_n\| \to \infty$. Then $\|N_n\| \to \infty$ and assume that $\frac{N_n}{\|N_n\|} \xrightarrow{\text{p}} N \in$ p-$B_F(x_0, y_0)_\infty \setminus \{0\}$. Dividing (13) by $(\frac{2t_n}{r_n})\|N_n\|$ we get in the limit $N(v, v) \notin -\text{cl } C$ which is absurd. □

Theorem 4.2 strictly contains Theorem 4.12 of [13] as a special case. We interpret the use of Theorem 4.2 by the following example.

Example 4.2. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $S = [0, +\infty)$, $C = \mathbb{R}^2_+$, $x_0 = 0$, $(y_0, z_0) = (0, 0) \in Y$ and

$$F(x) = \begin{cases} \{(y,z) \in \mathbb{R}^2 \mid y = x^2, \frac{3}{4}|x|^{\frac{4}{3}} \le z \le |x|^{\frac{4}{3}}\} & \text{if } x \ge 0, \\ \emptyset & \text{if } x < 0. \end{cases}$$

Then $(y_0, z_0) \in \mathrm{StrMin}_C F(x_0)$, $T(S, x_0) = S$ and, for a fixed $\alpha > 0$,

$$A^S_F(x_0, (y_0, z_0)) = \{0\} \times [0, 1] = \mathrm{cl}A^S_F(x_0, (y_0, z_0)),$$
$$B^S_F(x_0, (y_0, z_0)) = \{(1, z) \mid z > \alpha\},$$
$$\mathrm{cl}B^S_F(x_0, (y_0, z_0)) = \{(1, z) \mid z \ge \alpha\},$$
$$B^S_F(x_0, (y_0, z_0))_\infty = \{(0, z) \mid z \ge 0\}.$$

It is easy to check that, for all $v \in T(S, x_0) \setminus \{0\}$, one has

$$A^S_F(x_0, (y_0, z_0))v = \{(0, \beta v) \mid \beta \in [0, 1]\} \subseteq \mathrm{cl}\ C,$$
$$N(v, v) = (0, zv^2) \notin -\mathrm{cl}\ C,$$

$\forall N \in B^S_F(x_0, (y_0, z_0))_\infty \setminus \{0\}$, and

$$Mw + 2N(v, v) = (2v^2, 2zv^2) \notin -\mathrm{cl}C,$$

$\forall w \in T^2(S, x_0, v) \cap v^\perp = \{0\}$, $\forall M \in \mathrm{cl}A^S_F(x_0, (y_0, z_0)) = \{(0, 0)\}$, $\forall N \in \mathrm{cl}B^S_F(x_0, (y_0, z_0))$, and $T''(S, x_0, v) \cap v^\perp \setminus \{0\} = \emptyset$. Now that all assumptions of Theorem 4.2 are satisfied, $(x_0, (y_0, z_0)) \in$ LFE(2, P).

Summarizing it should be noted that each of the necessary conditions and sufficient conditions presented in this paper is an extension to the multivalued case of the corresponding result in [13] for the single-valued case. The results of [13] were shown in [13] to be sharper than the corresponding theorems in [15] and better in use than many recent results in the literature, since the assumptions are very relaxed.

References

[1] ALLALI, K., AMAHROQ, T. Second-order approximations and primal and dual necessary optimality conditions. *Optimization*, **40** (1997), 229-246.

[2] AMAHROQ, T., GADHI, N. Second-order optimality conditions for the extremal problem under inclusion constraints. *Journal of Mathematical Analysis and Applications*, **285** (2003), 74-85.

[3] AUBIN, J. P., FRANKOWSKA, H. *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[4] CAMBINI, A., MARTEIN, L., VLACH, M. Second-order tangent sets and optimality conditions. *Mathematica Japonica*, **49** (1999), 451-461.

[5] CLARKE, F. H. *Optimization and Nonsmooth Analysis*, Wiley Interscience, New York, 1983.

[6] DEMYANOV, V. F., RUBINOV, A. M. On quasidifferentiable functionals. *Soviet Mathematical Doklady*, **21** (1980), 14-17.

[7] GINCHEV, I., GUERRAGGIO, A., ROCCA, M. Second-order conditions for $C^{1,1}$ constrained vector optimization. *Mathematical Programming*, **104** (B) (2005), 389-405.

[8] JEYAKUMAR, V., LUC, D. T. Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization. *SIAM Journal on Control and Optimization*, **36** (1998), 1815-1832.

[9] JIMÉNEZ, B. Strict efficiency in vector optimization. *Journal of Mathematical Analysis and Applications*, **265** (2002), 264-284.

[10] JIMÉNEZ, B., NOVO, V. Optimality conditions in differentiable vector optimization via

second-order tangent sets. *Applied Mathematics and Optimization*, **49** (2004), 123-144.

[11] JOURANI, A., THIBAULT, L. Approximations and metric regularity in mathematical programming in Banach spaces. *Mathematics of Operations Research*, **18** (1992), 390-400.

[12] KHANH, P. Q., TUAN, N. D. First and second-order optimality conditions using approximations for nonsmooth vector optimization in Banach spaces. *Journal of Optimization Theory and Applications*, **130** (2006), 289-308.

[13] KHANH, P. Q., TUAN, N. D. First and second-order approximations as derivatives of mappings in optimality conditions for nonsmooth vector optimization. *Applied Mathematics and Optimization*, **58** (2008), 147-166.

[14] KHANH, P. Q., TUAN, N. D. Optimality conditions using approximations for nonsmooth vector optimization problems under general inequality constraints. *Journal of Convex Analysis*, **16** (2009), 169-186.

[15] KHANH, P. Q., TUAN, N. D. Optimality conditions for nonsmooth multiobjective optimization using Hadamard directional derivatives. *Journal of Optimization Theory and Applications*, **133** (2007), 341-357.

[16] KHANH, P. Q., TUAN, N. D. Approximations as generalized differentiability and optimality conditions without continuity assumptions in multivalued optimization. Inter. Confer. Nonlinnear Analysis in Engineering Mechanics Today, Hochiminh City, December 12-15, 2006.

[17] KHANH, P. Q., TUNG, N. M. Approximations as generalized derivatives: calculus and optimality conditions in multivalued optimization, submitted for publication.

[18] PENOT, J. P. Recent advances on second-order optimality conditions. *Optimization* (Nguyen, V. H., Strodiot, J. J., Tossings, P. eds), Springer, Berlin, 357-380, 2000.